

# Cosovereign Hopf algebras

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## Abstract

A sovereign monoidal category is an autonomous monoidal category endowed with the choice of an autonomous structure and an isomorphism of monoidal functors between the associated left and right duality functors. In this paper we define and study the algebraic counterpart of sovereign monoidal categories: cosovereign Hopf algebras. In this framework we find a categorical characterization of involutory Hopf algebras. We describe the universal cosovereign Hopf algebras and we also study finite-dimensional cosovereign Hopf algebras via the dimension theory provided by the sovereign structure.

AMS Classification: 16W30, 18D10.

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## 1 Introduction

Monoidal category theory played a central role in the discovery of new invariants of knots and links and in the development of the theory of quantum groups.

Let us recall that a tortile tensor category (or ribbon category) is a braided monoidal category ([13]), which is autonomous (ie every object has a left dual, and hence also a right dual) and admits a twist ([13, 23]) compatible with duality.

The connection with knot theory is certainly best resumed is the following coherence theorem by Shum ([23]): “*the category of framed tangles (or tangles on ribbons) is the free tortile (or ribbon) category generated by an object*”. This means that to any object in a tortile (or ribbon) category, one can associate an isotopy invariant of framed tangles ([23, 10, 27, 20]). We refer the reader to the book [14] for these topics.

A new structure for monoidal categories appeared in papers by Freyd and Yetter ([11, 34]). A sovereign structure on an autonomous monoidal category (ie with left and right duals) consists of the choice of a left and a right autonomous structure and a monoidal isomorphism between the associated left and right duality functors. A theorem of Deligne

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(proposition 2.11 in [34]) brought interest in sovereign structures: there is a twist on an autonomous braided monoidal category if and only if there is a sovereign structure on it.

Maltsiniotis ([18]) studied sovereign monoidal categories in their own right. He proposed a new equivalent definition (which avoids the choice of an autonomous structure) and he also showed that an axiom was redundant in [11, 34]. A sovereign structure is in fact the exact structure needed to define a trace theory.

Quantum groups and monoidal categories are linked by tannakian duality ([22, 6, 28]): the reconstruction of a Hopf algebra from its finite-dimensional comodules. The use of tannakian duality is clearly illustrated in [12]: to an additional categorical structure on the category of finite-dimensional comodules, one associates an additional algebraic structure on the Hopf algebra. For example if  $A$  is a Hopf algebra, then there is a braiding ([13]) on  $\text{Co}_f(A)$  (finite-dimensional  $A$ -comodules) if and only if there is a cobrading on  $A$ , ie a linear form on  $A \otimes A$  satisfying certain conditions (see the appendix or [14, 12]). In the same way if  $A$  is a cobarided Hopf algebra, the category  $\text{Co}_f(A)$  is balanced (there is a twist on it) if and only if there is a linear form  $\tau$  (called a cotwist) on  $A$  satisfying some conditions. Another example of the use of Tannaka duality is given by Street in [24]: the construction of the quantum double for any bialgebra (even infinite-dimensional).

In this paper we find the algebraic structure on Hopf algebras corresponding to sovereign structures. A sovereign character on a Hopf algebra  $A$  with bijective antipode is a *character*  $\Phi$  on  $A$  such that  $S^{-1} = \Phi * S * \Phi^{-1}$  ( $S$  is the antipode of  $A$  and  $*$  is the convolution product). A cosovereign Hopf algebra is a pair  $(A, \Phi)$  where  $A$  is a Hopf algebra with bijective antipode and  $\Phi$  is a sovereign character on  $A$ .

Let  $A$  be a Hopf algebra with a bijective antipode. We show that the category  $\text{Co}_f(A)$  admits a sovereign structure if and only if there is a sovereign character on  $A$  (in fact the main result, theorem 3.12, is more precise). We also obtain a categorical characterization of involutory Hopf algebras (Hopf algebras whose square of the antipode is equal to the identity): a Hopf algebra is involutory if and only if the category  $\text{Co}_f(A)$  admits a sovereign structure for which the forgetful functor is sovereign.

Back not too far from knot theory, the theorem of Deligne mentioned above states in particular the bijective correspondence of cotwists and sovereign characters for a cobarided Hopf algebra (we give a purely Hopf algebraic proof of this result in an appendix). But it is easier to check the existence of a sovereign character (since it is a character). Therefore a technical simplification is brought by sovereign structures in this context.

The end of the paper is devoted to the study of some examples. We describe the universal (or free) cosovereign Hopf algebras: every finite-type cosovereign Hopf algebra is a homomorphic quotient of one of them. These algebras are parameterized by an invertible matrix. When the base field is the field of complex numbers, they already appeared in a different context: they are the algebras of representative functions on the universal compact quantum groups defined by Van Daele and Wang ([30]). We also examine a class of examples closely related to the quantum groups  $SU(n)$  of Woronowicz ([33]) and show

that these Hopf algebras are cosovereign.

We also study finite-dimensional cosovereign Hopf algebras via the dimension theory provided by the sovereign structure (theorem 5.1). We show that if  $(A, \Phi)$  is a finite-dimensional cosovereign Hopf algebra over a field of characteristic zero whose internal dimension (ie the dimension computed in the sovereign monoidal category  $\text{Co}_f(A)$ ) is non-zero, then  $A$  is involutory (and therefore is semisimple and cosemisimple by [15, 16]).

The paper is organized as follows. In section 2 we review the basic definitions of monoidal category theory. In section 3 we introduce cosovereign (and sovereign) Hopf algebras and study their relations with sovereign structures for monoidal categories. Sections 4 and 5 are devoted to examples. In an appendix we examine the relations between sovereign characters and cotwists for cobraided Hopf algebras.

## Notations

Throughout this paper  $k$  will denote a commutative field. The category of finite-dimensional vector spaces will be denoted by  $\text{Vect}_f(k)$ .

We assume the reader to be familiar with the theory of Hopf algebras ([1, 25, 14]) and in particular we freely use convolution products.

Let  $A = (A, m, u, \Delta, \varepsilon, S)$  be a Hopf  $k$ -algebra. The multiplication will be denoted by  $m$ ,  $u : k \rightarrow A$  is the unit of  $A$ , while  $\Delta$ ,  $\varepsilon$  and  $S$  are respectively the comultiplication, the counit and the antipode of  $A$ .

If  $A$  is a Hopf algebra, the category of finite-dimensional right  $A$ -comodules will be denoted by  $\text{Co}_f(A)$  while the category of finite-dimensional left  $A$ -modules will be denoted by  $\text{Mod}_f(A)$ . We only consider right comodules and left modules.

## 2 Monoidal categories

The aim of this section is to recall the basic definitions of monoidal category theory and to fix some notations. We refer the reader to [17] or [13] for the general definitions of monoidal categories and monoidal functors. The material presented here is now classical and hence we will be a little concise. We only use strict monoidal categories: by MacLane's coherence theorem (see [13], 1.4 for a simple proof), every monoidal category is monoidally equivalent to a strict one. However the reconstruction theorem (2.12) we use deals with non strict monoidal categories. We will see that the general statement follows from the particular case of strict monoidal categories.

**Definition 2.1** *A monoidal category  $\mathcal{C} = (\mathcal{C}, \otimes, I)$  consists of a category  $\mathcal{C}$ , a functor  $\otimes : \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$  (called the tensor product of  $\mathcal{C}$ ) and an object  $I$  of  $\mathcal{C}$  (called the monoidal unit) such that for all objects  $X, Y, Z$  and all arrows  $f, g, h$  of  $\mathcal{C}$  we have:*

- 1) *associativity :  $(X \otimes Y) \otimes Z = X \otimes (Y \otimes Z)$  and  $(f \otimes g) \otimes h = f \otimes (g \otimes h)$ .*
- 2) *unit :  $I \otimes X = X = X \otimes I$  and  $1_I \otimes f = f = f \otimes 1_I$ .*

Let  $\mathcal{C} = (\mathcal{C}, \otimes, I)$  be a monoidal category. Then  $(\mathcal{C}^o, \otimes, I)$ ,  $(\mathcal{C}, \otimes^o, I)$  and  $(\mathcal{C}^o, \otimes^o, I)$  are also monoidal categories, where  $\mathcal{C}^o$  is the opposite category of  $\mathcal{C}$  and  $\otimes^o$  is the opposite tensor product of  $\mathcal{C}$  ( $X \otimes^o Y = Y \otimes X$ ).

**Definition 2.2** Let  $\mathcal{C}$  and  $\mathcal{D}$  be monoidal categories. A monoidal functor  $F = (F, \tilde{F})$  consists of a functor  $F : \mathcal{C} \rightarrow \mathcal{D}$ , a natural family of isomorphisms  $\tilde{F}_{X,Y} : F(X) \otimes F(Y) \xrightarrow{\sim} F(X \otimes Y)$  and an isomorphism  $\tilde{F}_I : I \xrightarrow{\sim} F(I)$  such that:

$$\begin{aligned} \tilde{F}_{X \otimes Y, Z} \circ (\tilde{F}_{X,Y} \otimes 1_{F(Z)}) &= \tilde{F}_{X, Y \otimes Z} \circ (1_{F(X)} \otimes \tilde{F}_{Y,Z}), \\ \tilde{F}_{I, X} \circ (\tilde{F}_I \otimes 1_{F(X)}) &= 1_{F(X)} = \tilde{F}_{X, I} \circ (1_{F(X)} \otimes \tilde{F}_I). \end{aligned}$$

A monoidal equivalence is a monoidal functor whose underlying functor is an equivalence of categories.

Let  $F, G : \mathcal{C} \rightarrow \mathcal{D}$  be monoidal functors. A morphism of monoidal functors  $u : F \rightarrow G$  is a natural transformation  $u : F \rightarrow G$  such that:

$$u_{X \otimes Y} \circ \tilde{F}_{X,Y} = \tilde{G}_{X,Y} \circ (u_X \otimes u_Y) \text{ and } u_I \circ \tilde{F}_I = \tilde{G}_I.$$

If  $u$  is also an isomorphism we write  $F \cong^{\otimes} G$ .

The set of morphisms between monoidal functors  $F$  and  $G$  will be denoted by  $\text{Hom}^{\otimes}(F, G)$ .

Let  $F : \mathcal{C} \rightarrow \mathcal{D}$  be a monoidal equivalence. By [22] there is a monoidal equivalence  $G : \mathcal{D} \rightarrow \mathcal{C}$  and isomorphisms of monoidal functors  $1_{\mathcal{C}} \cong^{\otimes} G \circ F$  and  $1_{\mathcal{D}} \cong^{\otimes} F \circ G$ .

Let  $F, G : \mathcal{C} \rightarrow \mathcal{D}$  be monoidal functors and let  $u : F \rightarrow G$  be a morphism of monoidal functors. Let  $K : \mathcal{D} \rightarrow \mathcal{E}$  be a monoidal functor. Then  $K(u) : KF \rightarrow KG$  is a morphism of monoidal functors. If  $u$  is an isomorphism, so is  $K(u)$ . Let  $K' : \mathcal{D} \rightarrow \mathcal{E}$  be another monoidal functor and let  $v : K \rightarrow K'$  be a morphism of monoidal functors. Then  $v_F : KF \rightarrow K'F$  is a morphism of monoidal functors. If  $v$  is an isomorphism, so is  $v_F$ .

**Definition 2.3** Let  $\mathcal{C} = (\mathcal{C}, \otimes, I)$  be a monoidal category and let  $X \in \text{ob}(\mathcal{C})$ . A left dual for  $X$  is a triplet  $({}^{\vee}X, \varepsilon_X, \eta_X)$  with  ${}^{\vee}X \in \text{ob}(\mathcal{C})$ ,  $\varepsilon_X : {}^{\vee}X \otimes X \rightarrow I$  and  $\eta_X : I \rightarrow X \otimes {}^{\vee}X$  are morphisms of  $\mathcal{C}$  such that:

$$(1_X \otimes \varepsilon_X) \circ (\eta_X \otimes 1_X) = 1_X \quad \text{and} \quad (\varepsilon_X \otimes 1_{{}^{\vee}X}) \circ (1_{{}^{\vee}X} \otimes \eta_X) = 1_{{}^{\vee}X}.$$

A right dual for  $X$  is a triplet  $(X^{\vee}, e_X, d_X)$  with  $X^{\vee} \in \text{ob}(\mathcal{C})$ ,  $e_X : X \otimes X^{\vee} \rightarrow I$  and  $d_X : I \rightarrow X^{\vee} \otimes X$  are morphisms of  $\mathcal{C}$  such that:

$$(1_{X^{\vee}} \otimes e_X) \circ (d_X \otimes 1_{X^{\vee}}) = 1_{X^{\vee}} \quad \text{and} \quad (e_X \otimes 1_X) \circ (1_X \otimes d_X) = 1_X.$$

Let  $X$  be an object of a monoidal category and suppose that  $X$  is endowed with a left dual. Then the functor  $X \otimes -$  admits a left adjoint  ${}^{\vee}X \otimes -$  (see [12], Section 9). Similarly, if there is a right dual for  $X$ , the functor  $X \otimes -$  admits a right adjoint which is  $X^{\vee} \otimes -$ . We inherit the unicity results of adjoint functors:

**Proposition 2.4** *Let  $X$  be an object in a monoidal category  $\mathcal{C}$ . Suppose that  $({}^\vee X, \varepsilon_X, \eta_X)$  and  $({}^*X, \varepsilon'_X, \eta'_X)$  are left duals for  $X$ . Then there is a unique isomorphism  $l_X : {}^\vee X \longrightarrow {}^*X$  such that:*

$$\varepsilon'_X \circ (l_X \otimes 1_X) = \varepsilon_X \quad (\text{and} \quad (1_X \otimes l_X) \circ \eta_X = \eta'_X).$$

*Suppose that  $(X^\vee, e_X, d_X)$  and  $(X^*, e'_X, d'_X)$  are right duals for  $X$ . Then there is a unique isomorphism  $r_X : X^\vee \longrightarrow X^*$  such that:*

$$e'_X \circ (1_X \otimes r_X) = e_X \quad (\text{and} \quad (r_X \otimes 1_X) \circ d_X = d'_X).$$

**Definition 2.5** *Let  $\mathcal{C}$  be a monoidal category. Let  $X, Y \in \text{ob}(\mathcal{C})$  and suppose that that  $X$  and  $Y$  admit a left dual (resp. a right dual). Let  $f \in \text{Hom}_{\mathcal{C}}(X, Y)$ . The left transpose of  $f$  (resp. right transpose of  $f$ ) is the morphism  ${}^t f : {}^\vee Y \longrightarrow {}^\vee X$  (resp.  $f^t : Y^\vee \longrightarrow X^\vee$ ) defined by:*

$$\begin{aligned} {}^t f &= (\varepsilon_Y \otimes 1_{\vee X}) \circ (1_{\vee Y} \otimes f \otimes 1_{\vee X}) \circ (1_{\vee Y} \otimes \eta_X) \\ (\text{resp. } f^t &= (1_{X^\vee} \otimes e_Y) \circ (1_{X^\vee} \otimes f \otimes 1_{Y^\vee}) \circ (d_Y \otimes 1_{Y^\vee})). \end{aligned}$$

It is easily seen that  ${}^t f$  (resp.  $f^t$ ) is the only morphism satisfying  $\varepsilon_X \circ ({}^t f \otimes 1_X) = \varepsilon_Y \circ (1_{\vee Y} \otimes f)$  or  $(1_Y \otimes {}^t f) \circ \eta_Y = (f \otimes 1_{\vee X}) \circ \eta_X$  (resp.  $e_X \circ (1_X \otimes f^t) = e_Y \circ (1_{\vee Y} \otimes f)$  or  $(f^t \otimes 1_Y) \circ d_Y = (1_{X^\vee} \otimes f) \circ d_X$ ).

**Definition 2.6** *A monoidal category is said to be left autonomous (resp. right autonomous ; autonomous) if every object has a left dual (resp. right dual ; resp left and right duals).*

**Definition 2.7** *Let  $\mathcal{C}$  be a left (resp. right) autonomous monoidal category. A left (resp. right) autonomous structure on  $\mathcal{C}$  is the choice for every object of a left dual  $({}^\vee X, \varepsilon_X, \eta_X)$  (resp. a right dual  $(X^\vee, e_X, d_X)$ ) such that  ${}^\vee I = I$  and  $\varepsilon_I = \eta_I = 1_I$  (resp.  $I^\vee = I$  and  $e_I = d_I = 1_I$ ). An autonomous structure on an autonomous monoidal category  $\mathcal{C}$  consists of a left autonomous structure on  $\mathcal{C}$  and a right autonomous structure on  $\mathcal{C}$ .*

**Definition 2.8** *Let  $\mathcal{C} = (\mathcal{C}, \otimes, I)$  be an autonomous monoidal category. Choose an autonomous structure on  $\mathcal{C}$ . We get two monoidal functors:*

$\mathbf{D}_l : (\mathcal{C}, \otimes, I) \longrightarrow (\mathcal{C}^\circ, \otimes^\circ, I)$  defined by  $\mathbf{D}_l(X) = {}^\vee X$  and  $\mathbf{D}_l(f) = {}^t f$ ,  
 $\mathbf{D}_r : (\mathcal{C}, \otimes, I) \longrightarrow (\mathcal{C}^\circ, \otimes^\circ, I)$ , defined by  $\mathbf{D}_r(X) = X^\vee$  and  $\mathbf{D}_r(f) = f^t$ ,  
*called the left duality functor and the right duality functor respectively.*

Another choice of autonomous structure would lead to monoidal functors  $\mathbf{D}'_l$  and  $\mathbf{D}'_r$  with  $\mathbf{D}'_l \cong^{\otimes} \mathbf{D}_l$  and  $\mathbf{D}'_r \cong^{\otimes} \mathbf{D}_r$  (see proposition 2.4). Thus the choice of an autonomous structure is just a convenient way to define the duality functors.

Let us remark that the duality functors can also be seen as monoidal functors  $(\mathcal{C}^\circ, \otimes^\circ, I) \longrightarrow (\mathcal{C}, \otimes, I)$ . There are isomorphisms of monoidal functors (see proposition 2.4):

$$h : 1_{\mathcal{C}} \cong^{\otimes} \mathbf{D}_l \circ \mathbf{D}_r \quad (2.8.1) \quad \kappa : 1_{\mathcal{C}} \cong^{\otimes} \mathbf{D}_r \circ \mathbf{D}_l \quad (2.8.2)$$

Let  $\mathcal{C}$  and  $\mathcal{D}$  be autonomous monoidal categories endowed with autonomous structures and let  $F : \mathcal{C} \longrightarrow \mathcal{D}$  be a monoidal functor. Then there are isomorphisms of monoidal functors (see proposition 2.4 again):

$$l : \mathbf{D}_l \circ F \cong^{\otimes} F \circ \mathbf{D}_l \quad (2.8.3) \quad r : \mathbf{D}_r \circ F \cong^{\otimes} F \circ \mathbf{D}_r \quad (2.8.4)$$

**Example 2.9** We briefly describe the autonomous structure on  $\text{Vect}_f(k)$  and  $\text{Co}_f(A)$  where  $A$  a Hopf algebra with bijective antipode. Let  $V$  be a finite-dimensional vector space. Let  ${}^\vee V = V^\vee = V^* = \text{Hom}(V, k)$ . It is well known that this procedure, with classical evaluation and coevaluation maps, defines an autonomous structure on  $\text{Vect}_f(k)$ . Now let  $V$  be a finite-dimensional  $A$ -comodule with coaction  $\alpha_V : V \longrightarrow V \otimes A$  such that  $\alpha_V(v_i) = \sum_j v_j \otimes a_{ji}$  for some basis  $(v_i)$  of  $V$ . Now let  ${}^\vee V$  (resp.  $V^\vee$ ) be the  $A$ -comodule whose underlying vector space is  $V^*$  and whose coaction  $\alpha_{{}^\vee V} : {}^\vee V \longrightarrow {}^\vee V \otimes A$  (resp.  $\alpha_{V^\vee} : V^\vee \longrightarrow V^\vee \otimes A$ ) is defined by  $\alpha_{{}^\vee V}(v_i^*) = \sum_j v_j^* \otimes S(a_{ij})$  (resp.  $\alpha_{V^\vee}(v_i^*) = \sum_j v_j^* \otimes S^{-1}(a_{ij})$ ) where  $(v_i^*)$  is the dual basis of  $(v_i)$ . Then  ${}^\vee V$  and  $V^\vee$ , with classical evaluation and coevaluation maps, are left and right duals for  $V$ .

We now proceed to describe the tannakian reconstruction theorems. First we need the following definition as given in [12]:

**Definition 2.10** Let  $\mathcal{C}$  be a small category and let  $F : \mathcal{C} \longrightarrow \text{Vect}_f(k)$  be a functor. Let  $\mathcal{N}$  be the subspace of  $\bigoplus_{X \in \text{ob } \mathcal{C}} \text{End}(F(X))$  generated by the expressions  $g \circ F(f) - F(f) \circ g$ , where  $f \in \text{Hom}_{\mathcal{C}}(X, Y)$  and  $g \in \text{Hom}(F(Y), F(X))$ . We define

$$\text{End}^\vee(F) = \bigoplus_{X \in \text{ob } \mathcal{C}} \text{End}(F(X)) / \mathcal{N}$$

Let  $g \in \text{End}(F(X))$ . We denote it by  $[X, g]$  as an element of  $\text{End}^\vee(F)$ .

**Theorem 2.11** Let  $\mathcal{C}$  be a small category and let  $F : \mathcal{C} \longrightarrow \text{Vect}_f(k)$  be a functor.  
i) The vector space  $\text{End}^\vee(F)$  is a coalgebra with coproduct given by

$$\Delta([X, \phi \otimes x]) = \sum_i [X, \phi \otimes v_i] \otimes [X, v_i^* \otimes x],$$

where  $X$  is an object of  $\mathcal{C}$ ,  $\phi \in F(X)^*$ ,  $x \in F(X)$  and  $(v_i)$  is a basis of  $F(X)$ . The counit is given by  $\varepsilon([X, g]) = \text{Tr}(g)$ . There is a linear isomorphism

$$\text{End}(F) \longrightarrow \text{End}^\vee(F)^*$$

(where  $\text{End}(F)$  is the algebra of endomorphisms of the functor  $F$ ) which to  $u \in \text{End}(F)$ , associates the linear form  $f_u$  defined by  $f_u([X, g]) = \text{Tr}(u_X \circ g)$ .

ii) Suppose now that  $\mathcal{C} = (\mathcal{C}, \otimes, I)$  is a monoidal category and that  $F = (F, \tilde{F})$  is a monoidal functor. Then there is a bialgebra structure on  $\text{End}^\vee(F)$  whose product is given by

$$[X, g].[Y, h] = [X \otimes Y, \tilde{F}_{X,Y} \circ (g \otimes h) \circ \tilde{F}_{X,Y}^{-1}]$$

and whose unit is  $[I, 1]$ .

iii) Suppose that  $\mathcal{C}$  is a left autonomous monoidal category and that  $F$  is a monoidal functor. Then the bialgebra  $\text{End}^\vee(F)$  is a Hopf algebra with antipode  $S$  given by

$$S([X, g]) = [{}^\vee X, l_X \circ {}^t g \circ l_X^{-1}]$$

( $l_X$  is from 2.8.3). The linear isomorphism of i) induces a bijection

$$\text{Aut}^\otimes(F) \longrightarrow \text{Hom}_{k\text{-alg}}(\text{End}^\vee(F), k)$$

where  $\text{Aut}^\otimes(F)$  denotes the set of endomorphisms of the monoidal functor  $F$  (which are automorphisms). If furthermore  $\mathcal{C}$  is autonomous, then the antipode of  $\text{End}^\vee(F)$  is bijective, with inverse given by

$$S^{-1}([X, g]) = [X^\vee, r_X \circ g^t \circ r_X^{-1}]$$

( $r_X$  is from 2.8.4).

The proof of this theorem can be found in [12] for example. Statements i) and ii) are due to Saavedra [22]. Statement iii) is due to Ulbrich [28]. This theorem is the framework for the following reconstruction theorem, whose proof can be found in many places and at different levels of generality ([22, 6, 5, 12]).

**Theorem 2.12** i) Let  $\mathcal{C}$  be an autonomous monoidal (small) category and let  $F : \mathcal{C} \longrightarrow \text{Vect}_f(k)$  be a monoidal functor. Then  $F$  factorizes through a monoidal functor  $\overline{F} : \mathcal{C} \longrightarrow \text{Co}_f(\text{End}^\vee(F))$  followed by the forgetful functor.

ii) Let  $\mathcal{C}$  be a (small)  $k$ -tensorial autonomous monoidal category and let  $F : \mathcal{C} \longrightarrow \text{Vect}_f(k)$  be a fibre functor. Then the above functor  $\overline{F}$  is a monoidal equivalence of categories.

iii) Let  $\mathcal{C} = \text{Co}_f(A)$  be the category of right  $A$ -comodules where  $A$  is a Hopf algebra with bijective antipode and let  $F$  be the forgetful functor. Then the Hopf algebras  $A$  and  $\text{End}^\vee(F)$  are isomorphic.

We have just included statement ii) for the sake of completeness. We refer the reader to [5] for the precise definitions.

**Remark 2.13** In theorem 2.11 and 2.12, the encountered monoidal categories must not be assumed to be strict for useful applications. This is clear from statement iii) in theorem 2.12. However, assume that theorem 2.11 has been proved for strict monoidal categories. Let  $\mathcal{C}$  be a (non-strict) monoidal category and let  $F : \mathcal{C} \longrightarrow \text{Vect}_f(k)$  be a monoidal functor. Let  $St(\mathcal{C})$  be a strict monoidal category with a monoidal equivalence  $i : St(\mathcal{C}) \longrightarrow \mathcal{C}$ . Then the coalgebras  $\text{End}^\vee(F)$  and  $\text{End}^\vee(F \circ i)$  are easily seen to be isomorphic and therefore statements ii) and iii) of theorem 2.11 follow in the general case (and the same formulas hold); In the same way, statements i) and ii) of theorem 2.12 follow from the strict monoidal case.

Let  $A$  be a Hopf algebra. Assume that there is an additional categorical structure on  $\text{Co}_f(A)$  : we would like to translate it into an additional algebraic structure on  $A$ . It is clear from the precedent discussion that it is sufficient to make this translation for  $\text{End}^\vee(F)$ , where  $F : \mathcal{C} \longrightarrow \text{Vect}_f(k)$  is a monoidal functor and  $\mathcal{C}$  is a strict monoidal category.

**Remark 2.14** When  $k$  is a ring theorem 2.11 is still valid in some sense: one has to replace  $\text{Vect}_f(k)$  by  $\text{Proj}_f(k)$ , the category of finitely generated projective  $k$ -modules. Let us note that if  $\mathcal{C}$  is autonomous, a monoidal functor  $F : \mathcal{C} \longrightarrow \text{Mod}(k)$  take values in  $\text{Proj}_f(k)$  ([6], 2.6).

### 3 Sovereign monoidal categories and cosovereign Hopf algebras

In this section we give our main result: the correspondence between categorical and algebraic sovereign structures.

**Definition 3.1** Let  $\mathcal{C} = (\mathcal{C}, \otimes, I)$  be an autonomous monoidal category. A sovereign structure  $\varphi$  on  $\mathcal{C}$  consists of an autonomous structure on  $\mathcal{C}$  and an isomorphism of monoidal functors  $\varphi : \mathbf{D}_r \cong^\otimes \mathbf{D}_l$  for the duality functors. A sovereign monoidal category  $\mathcal{C} = (\mathcal{C}, \otimes, I, \varphi)$  is an autonomous monoidal category endowed with a sovereign structure.

**Remark 3.2** Let  $\mathcal{C} = (\mathcal{C}, \otimes, I, \varphi)$  be a sovereign monoidal category. Let us choose another autonomous structure on  $\mathcal{C}$  and let us denote by  $\mathbf{D}'_l$  and  $\mathbf{D}'_r$  the associated duality functors. Then there is an isomorphism  $\mathbf{D}'_r \cong^\otimes \mathbf{D}_r \varphi \cong^\otimes \mathbf{D}_l \cong^\otimes \mathbf{D}'_l$ . In this way we get another sovereign structure  $\varphi'$  on  $\mathcal{C}$ . This fact suggests that we possibly could give a definition of sovereign structure which does not depend on the choice of an autonomous structure. This is done by Maltsiniotis in definition 3.1.2 of [18]. He shows in theorem 3.2.2 that his definition coincides with definition 3.1 given here.

In the earlier definition by Freyd and Yetter ([11, 34]) there was another axiom, which was shown to be redundant by Maltsiniotis (proposition 3.2.3 in [18]).



**Definition 3.3** Let  $(\mathcal{C}, \otimes, I, \varphi)$  and  $(\mathcal{D}, \otimes, I, \psi)$  be sovereign monoidal categories. A monoidal functor  $F = (F, \tilde{F}) : (\mathcal{C}, \otimes, I) \longrightarrow (\mathcal{D}, \otimes, I)$  is said to be sovereign if we have  $F(\varphi) \circ r = l \circ \psi_F$  where  $l$  and  $r$  are the isomorphisms of (2.8.3) and (2.8.4).

**Example 3.4** The category of diagrams ([34]) is the free sovereign monoidal category generated by an object ([11]).

An autonomous braided monoidal category admits a sovereign structure if and only there is a twist on it ([34], see also [18]).

The following lemmas will be useful :

**Lemma 3.5** Let  $\mathcal{C} = (\mathcal{C}, \otimes, I, \varphi)$  and  $(\mathcal{D}, \otimes, I, \psi)$  be sovereign monoidal categories and let  $F = (F, \tilde{F}) : (\mathcal{C}, \otimes, I) \longrightarrow (\mathcal{D}, \otimes, I)$  be a monoidal functor. There is an element  $u^{\varphi, \psi} \in \text{Aut}^{\otimes}(F)$  defined by

$$u^{\varphi, \psi} = \kappa_F^{-1} \circ \mathbf{D}_r(l^{-1} \circ F(\varphi) \circ r \circ \psi_F^{-1}) \circ \kappa_F$$

(where  $\kappa$ ,  $l$  and  $r$  are the isomorphisms of (2.8.2), (2.8.3) and (2.8.4)). If furthermore  $F$  is sovereign, then  $u^{\varphi, \psi} = 1_F$ .

**Proof.** The element  $u^{\varphi, \psi}$  of  $\text{End}(F)$  just defined is the composition of isomorphisms of monoidal functors. Hence  $u^{\varphi, \psi}$  is an automorphism of the monoidal functor  $F$ . If  $F$  is a sovereign functor, then  $l^{-1} \circ F(\varphi) \circ r \circ \psi_F^{-1}$  is the identity morphism of  $\mathbf{D}_l \circ F$ , and thus  $u^{\varphi, \psi} = 1_F$ .  $\square$

**Lemma 3.6** Let  $\mathcal{C} = (\mathcal{C}, \otimes, I, \varphi)$  and  $\mathcal{D} = (\mathcal{D}, \otimes, I, \psi)$  be sovereign monoidal categories and let  $F : (\mathcal{C}, \otimes, I, \varphi) \longrightarrow (\mathcal{D}, \otimes, I, \psi)$  be a sovereign monoidal functor. Suppose that we have another choice of autonomous structure on  $\mathcal{D}$  and let  $\psi'$  the associated sovereign structure of remark 3.2. Then  $F : (\mathcal{C}, \otimes, I, \varphi) \longrightarrow (\mathcal{D}, \otimes, I, \psi')$  is still a sovereign functor.

**Proof.** Let  $\mathbf{D}'_r$  and  $\mathbf{D}'_l$  be the duality functors on  $\mathcal{D}$ . Let  $\alpha : \mathbf{D}'_r \cong^{\otimes} \mathbf{D}_r$  and  $\beta : \mathbf{D}'_l \cong^{\otimes} \mathbf{D}_l$  be the isomorphisms given by proposition 2.4:  $\psi' = \beta^{-1} \circ \psi \circ \alpha$ . Let  $r' : \mathbf{D}'_r \circ F \cong^{\otimes} F \circ \mathbf{D}_r$ ,  $l' : \mathbf{D}'_l \circ F \cong^{\otimes} F \circ \mathbf{D}_l$ ,  $r : \mathbf{D}_r \circ F \cong^{\otimes} F \circ \mathbf{D}_r$ ,  $l : \mathbf{D}_l \circ F \cong^{\otimes} F \circ \mathbf{D}_l$  as in (2.8.3) and (2.8.4). Then we have  $r' = r \circ \alpha_F$  and  $l' = l \circ \beta_F$  (since they verify the equations in proposition 2.4). Thus we have  $F(\varphi) \circ r' = F(\varphi) \circ r \circ \alpha_F = l \circ \psi_F \circ \alpha_F = l' \circ \psi'_F$ .  $\square$

We now come into the heart of the subject:

**Definition 3.7** Let  $A$  be a Hopf algebra with bijective antipode. A sovereign character on  $A$  is a character  $\Phi$  on  $A$  such that  $S^{-1} = \Phi * S * \Phi^{-1}$ . A cosovereign Hopf algebra is a pair  $(A, \Phi)$  where  $A$  is a Hopf algebra with bijective antipode and  $\Phi$  is a sovereign character on  $A$ .

**Remark 3.8** We could have defined a cosovereign Hopf algebra as a pair  $(A, \Phi)$  consisting of a Hopf algebra  $A$  and a character  $\Phi$  on  $A$  such that  $S^2 = \Phi^{-1} * id * \Phi$ . The bijectivity of the antipode follows immediately. We have chosen this definition since it is more natural from the categorical viewpoint.

There is an immediate dual definition:

**Definition 3.9** Let  $A$  be a Hopf algebra with bijective antipode. A sovereign element of  $A$  is a group-like element  $\Phi$  such that  $S^{-1}(a) = \Phi S(a) \Phi^{-1}$  for all  $a \in A$ . A sovereign Hopf algebra is a pair  $(A, \Phi)$  where  $A$  is a Hopf algebra with bijective antipode and  $\Phi$  is a sovereign element of  $A$ .

It is clear that if  $(A, \Phi)$  is a cosovereign (resp. sovereign) Hopf algebra then  $(A^0, \Phi)$  is a sovereign (resp. cosovereign) Hopf algebra.

**Proposition 3.10** Let  $(A, \Phi)$  be a cosovereign Hopf algebra. Then the sovereign character  $\Phi$  defines a sovereign structure on  $\text{Co}_f(A)$ .

**Proof.** We use the autonomous structure on  $\text{Co}_f(A)$  described in example 2.9. Let  $V$  be a finite-dimensional  $A$ -comodule with coaction  $\alpha_V$ . We define a linear map  $\varphi_V : V^\vee \longrightarrow {}^\vee V$  as follows:  $\varphi_V = (1_{V^\vee} \otimes \Phi^{-1}) \circ \alpha_{V^\vee}$  (we use the fact  ${}^\vee V = V^\vee$  as vector space). For the convenience of the reader, let us describe  $\varphi_V$  explicitly. If  $(v_i)$  is a basis of  $V$  such that  $\alpha_V(v_i) = \sum_j v_j \otimes a_{ji}$ , we have  $\varphi_V(v_i^*) = \sum_j \Phi(a_{ij}) v_j^*$  for the dual basis  $(v_i^*)$  (recall that  $\Phi^{-1} = \Phi \circ S$ ). It is easy to check that  $\varphi_V$  is a map of comodules, because  $\Phi * S = S^{-1} * \Phi$ . Also it is easily seen that if  $f : V \longrightarrow W$  is a map of comodules, then  ${}^t f \circ \varphi_W = \varphi_V \circ f^t$ . In this way we get a bijective natural transformation  $\varphi : \mathbf{D}_r \longrightarrow \mathbf{D}_l$ . Finally it is easy to check that  $\varphi$  is a morphism of monoidal functors since  $\Phi$  is a character of  $A$  and therefore we have defined a sovereign structure on  $\text{Co}_f(A)$ .  $\square$

**Corollary 3.11** Let  $(A, \Phi)$  be a sovereign Hopf algebra. Then there is a sovereign structure on  $\text{Mod}_f(A)$ .

**Proof.** Let  $A^0$  be the dual Hopf algebra of  $A$ . The categories  $\text{Mod}_f(A)$  and  $\text{Co}_f(A^0)$  are monoidally equivalent and  $(A^0, \Phi)$  is a cosovereign Hopf algebra.  $\square$

We want a converse of proposition 3.10. We obtain a more precise result in the reconstruction setting:

**Theorem 3.12** i) Let  $\mathcal{C} = (\mathcal{C}, \otimes, I, \varphi)$  be a sovereign monoidal category and let  $F : \mathcal{C} \longrightarrow \text{Vect}_f(k)$  be a monoidal functor. Then there is a sovereign character on the Hopf algebra  $\text{End}^\vee(F)$ .

ii) If the monoidal functor  $F$  is sovereign (with respect to any sovereign structure on  $\text{Vect}_f(k)$ ), then the square of the antipode of  $\text{End}^\vee(F)$  is equal to the identity.

**Proof.** i) Let us endow  $\text{Vect}_f(k)$  with the autonomous structure of example 2.9 and with the isomorphism of monoidal functors  $\psi : \mathbf{D}_r \longrightarrow \mathbf{D}_l$  which is the identity. It is easy to see that  $\psi$  is unique for this choice of autonomous structure. We get in this way a sovereign structure on  $\text{Vect}_f(k)$ . Let us consider the element  $u^{\varphi, \psi} \in \text{Aut}^{\otimes}(F)$  of lemma 3.5. By theorem 2.11.iii)  $u^{\varphi, \psi}$  gives rise to a character  $\Phi$  on  $\text{End}^{\vee}(F)$ . Let us describe  $\Phi$ . Let  $X \in \text{ob}(\mathcal{C})$  and  $f \in \text{End}(F(X))$ . Then

$$\begin{aligned} \Phi([X, f]) &= \text{Tr}(f \circ u_X^{\varphi, \psi}) \\ &= \text{Tr}(f \circ \kappa_{F(X)}^{-1} \circ (l_X^{-1} \circ F(\varphi_X) \circ r_X \circ \psi_{F(X)}^{-1})^t \circ \kappa_{F(X)}) \\ &= \text{Tr}((l_X^{-1} \circ F(\varphi_X) \circ r_X \circ \psi_{F(X)}^{-1})^t \circ \kappa_{F(X)} \circ f \circ \kappa_{F(X)}^{-1}) \\ &= \text{Tr}((l_X^{-1} \circ F(\varphi_X) \circ r_X \circ \psi_{F(X)}^{-1})^t \circ ({}^t f)^t) \\ &= \text{Tr}({}^t f \circ l_X^{-1} \circ F(\varphi_X) \circ r_X \circ \psi_{F(X)}^{-1}) \end{aligned}$$

and  $\Phi^{-1}([X, f]) = \text{Tr}({}^t f \circ \psi_{F(X)} \circ r_X^{-1} \circ F(\phi_X)^{-1} \circ l_X)$ . We want to show that  $S^{-1} = \Phi * S * \Phi^{-1}$ . Let  $\gamma_X = l_X^{-1} \circ F(\varphi_X) \circ r_X \circ \psi_{F(X)}^{-1}$  and  $\beta_X = F(\varphi_X) \circ r_X \circ \psi_{F(X)}^{-1}$ . We have

$$\begin{aligned} S^{-1}([X, f]) &= [X^{\vee}, r_X \circ f^t \circ r_X^{-1}] \\ &= [{}^{\vee} X, F(\varphi_X) \circ r_X \circ \psi_X^{-1} \circ {}^t f \circ \psi_{F(X)} \circ r_X^{-1} \circ F(\varphi_X^{-1})] \\ &= [{}^{\vee} X, \beta_X \circ {}^t f \circ \beta_X^{-1}]. \end{aligned}$$

It is sufficient to work with rank one operators. Let  $(e_i)$  be a basis of  $F(X)$  with dual basis  $(e_i^*)$  and bidual basis  $(e_i^{**})$ . Then we have

$$S^{-1}([X, e_i^* \otimes e_j]) = [{}^{\vee} X, \beta_X \circ (e_j^{**} \otimes e_i^*) \circ \beta_X^{-1}] = [{}^{\vee} X, (e_j^{**} \circ \beta_X^{-1}) \otimes (\beta_X(e_i^*))].$$

On the other hand

$$\begin{aligned} &\Phi * S * \Phi^{-1}([X, e_i^* \otimes e_j]) \\ &= \sum_{p, q} \text{Tr}((e_p^{**} \otimes e_i^*) \circ \gamma_X) \text{Tr}((e_j^{**} \otimes e_q^*) \circ \gamma_X^{-1}) [{}^{\vee} X, l_X \circ (e_q^{**} \otimes e_p^*) \circ l_X^{-1}] \\ &= \sum_{p, q} e_p^{**}(\gamma_X(e_i^*)) e_j^{**}(\gamma_X^{-1}(e_q^*)) [{}^{\vee} X, (e_q^{**} \circ l_X^{-1}) \otimes (l_X(e_p^*))] \\ &= \left[ {}^{\vee} X, \left( \sum_q e_j^{**}(\beta_X^{-1} \circ l_X(e_q^*)) e_q^{**} \circ l_X^{-1} \right) \otimes \left( \sum_p e_p^{**}(\beta_X \circ l_X^{-1}(e_i^*)) l_X(e_p^*) \right) \right] \\ &= [{}^{\vee} X, (e_j^{**} \circ \beta_X^{-1}) \otimes (\beta_X(e_i^*))] = S^{-1}([X, e_i^* \otimes e_j]). \end{aligned}$$

Therefore  $S^{-1} = \Phi * S * \Phi^{-1}$  and the proof of i) is complete.

ii) Let us first suppose that  $F$  is sovereign with respect to the sovereign structure defined on  $\text{Vect}_f(k)$  in i). Then  $u^{\varphi, \psi} = 1_F$  by lemma 3.5 and  $S^{-1} = S$  by i). The general

statement follows now from lemma 3.6 and the fact that the isomorphism of monoidal functors  $\psi : \mathbf{D}_r \longrightarrow \mathbf{D}_l$  used in the proof of i) is unique with respect to the choice of the autonomous structure of example 2.9.  $\square$

**Remark 3.13** There is a unique sovereign structure on  $\text{Vect}_f(k)$  in the sense of [18].

**Remark 3.14** Theorem 3.12 is still valid if  $k$  is a ring (see remark 2.14).

**Corollary 3.15** *i) Let  $A$  be a Hopf algebra with bijective antipode. Then there is a sovereign character on  $A$  if and only if there is a sovereign structure on  $\text{Co}_f(A)$ .  
ii) Let  $A$  be a Hopf algebra. Then the square of the antipode of  $A$  is equal to the identity if and only if  $\text{Co}_f(A)$  is autonomous and admits a sovereign structure for which the forgetful functor is sovereign (with respect to any sovereign structure on  $\text{Vect}_f(k)$ ).*

**Proof.** Statement i) follows from theorem 3.12.i) and theorem 2.12.iii). Statement ii) follows from theorem 3.12.ii) and theorem 2.12.iii).  $\square$

The requirement  $S \circ S = id_A$  for a Hopf algebra (such Hopf algebras are often called involutory) was natural for the purpose of adapting results from group theory to a more general setting (see for example a generalization of Mashke's theorem in [15]). However there was no categorical notion on which this notion could rely. Corollary 3.15 gives a characterization of involutory Hopf algebras from the categorical viewpoint.

We end this section with a quick look at dimension theory. Maltsiniotis has shown in [18] that a sovereign structure is exactly the one needed to define a trace theory.

**Definition 3.16** *Let  $\mathcal{C} = (\mathcal{C}, \otimes, I, \varphi)$  be a sovereign monoidal category and let  $X \in \text{ob}(\mathcal{C})$ . The left (resp. right) dimension of  $X$  is the element of  $\text{End}(X)$  defined by*

$$\dim_{\varphi}^l(X) = \varepsilon_X \circ (\varphi_X \otimes 1_X) \circ d_X$$

$$(\text{Resp. } \dim_{\varphi}^r(X) = e_X \circ (1_X \otimes \varphi_X^{-1}) \circ \eta_X)$$

*If  $(A, \Phi)$  is a cosovereign Hopf algebra (resp. sovereign Hopf algebra) the left and right dimension of an object  $V$  of  $\text{Co}_f(A)$  (resp.  $\text{Mod}_f(A)$ ) are denoted by  $\dim_{\Phi}^l(V)$  and  $\dim_{\Phi}^r(V)$ .*

Let us note that the left and right dimensions may not coincide.

Let  $(A, \Phi)$  be a cosovereign Hopf algebra and let  $V$  be a finite-dimensional  $A$ -comodule with basis  $(v_i)$  such that  $\alpha_V(v_i) = \sum_j v_j \otimes a_{ji}$ . Then  $\dim_{\Phi}^l(V) = \sum_i \Phi(a_{ii})$  and  $\dim_{\Phi}^r(V) = \sum_i \Phi(S(a_{ii}))$ . Let  $(A, \Phi)$  be a sovereign Hopf algebra and let  $V$  be a finite-dimensional  $A$ -module. We have  $\dim_{\Phi}^l(V) = \text{Tr}(\Phi)$  and  $\dim_{\Phi}^r(V) = \text{Tr}(\Phi^{-1})$  where  $\Phi$  and  $\Phi^{-1}$  are considered as operators on  $V$ .

**Proposition 3.17** (see [18], corollary 3.5.25). Let  $\mathcal{C} = (\mathcal{C}, \otimes, I, \varphi)$  be a sovereign monoidal category and let  $X$  and  $Y$  be objects of  $\mathcal{C}$ .

i)  $\dim_{\varphi}^l(I) = 1_I = \dim_{\varphi}^r(I)$

ii) If  $X \cong Y$  then  $\dim_{\varphi}^l(X) = \dim_{\varphi}^l(Y)$  and  $\dim_{\varphi}^r(X) = \dim_{\varphi}^r(Y)$

iii)  $\dim_{\varphi}^l({}^{\vee}X) = \dim_{\varphi}^r(X) = \dim_{\varphi}^l(X^{\vee})$  ;  $\dim_{\varphi}^r({}^{\vee}X) = \dim_{\varphi}^l(X) = \dim_{\varphi}^r(X^{\vee})$ .

iv) If  $\text{End}(I)$  is central ( $u \otimes f = f \otimes u$  for all morphisms  $f$  of  $\mathcal{C}$  and all  $u \in \text{End}(I)$ ) then  $\dim_{\varphi}^l(X \otimes Y) = \dim_{\varphi}^l(X)\dim_{\varphi}^l(Y)$  and  $\dim_{\varphi}^r(X \otimes Y) = \dim_{\varphi}^r(X)\dim_{\varphi}^r(Y)$ .

## 4 Universal cosovereign Hopf algebras

In this section we introduce the universal cosovereign Hopf algebras and study some of their properties. By universal we mean that every finite type cosovereign Hopf algebra is a homomorphic quotient of one of them (a quantum subgroup in the language of quantum groups).

**Notations.** We will use matrix notations. If  $u = (u_{ij})$  is a  $n \times n$  matrix with values in any algebra, the transpose matrix of  $u$  will be denoted by  ${}^t u$ .

**Definition 4.1** Let  $F \in GL_n(k)$ . The algebra  $H(F)$  is the universal algebra with generators  $(u_{ij})_{1 \leq i, j \leq n}$ ,  $(v_{ij})_{1 \leq i, j \leq n}$  and relations:

$$u {}^t v = {}^t v u = 1 \quad ; \quad v F {}^t u F^{-1} = F {}^t u F^{-1} v = 1$$

The algebras  $H(F)$  are closely related to the universal compact quantum groups of Van Daele and Wang [30]. We go back to this subject later in the section.

**Theorem 4.2** Let  $F \in GL_n(k)$ . Then  $H(F)$  is a Hopf algebra:

$$\text{with comultiplication } \Delta(u_{ij}) = \sum_k u_{ik} \otimes u_{kj} \quad ; \quad \Delta(v_{ij}) = \sum_k v_{ik} \otimes v_{kj},$$

$$\text{with counit } \varepsilon(u_{ij}) = \delta_{ij} = \varepsilon(v_{ij}),$$

$$\text{with antipode } S(u) = {}^t v \quad ; \quad S(v) = F {}^t u F^{-1}.$$

The antipode of  $H(F)$  is bijective and its inverse is given by  $S^{-1}(u) = {}^t F {}^t v {}^t F^{-1}$  and  $S^{-1}(v) = {}^t u$ . There is a sovereign character  $\Phi_F$  on  $H(F)$  defined by  $\Phi_F(u) = {}^t F$  and  $\Phi_F(v) = F^{-1}$  and hence  $(H(F), \Phi_F)$  is a cosovereign Hopf algebra.

If  $A$  is a Hopf algebra with bijective antipode and if  $V$  is a finite-dimensional  $A$ -comodule with coaction  $\alpha_V : V \longrightarrow V \otimes A$  such that  $V^{\vee} \cong {}^{\vee} V$ , then there is a matrix  $F \in GL_n(k)$  ( $n = \dim(V)$ ), a coaction  $\beta_V : V \longrightarrow V \otimes H(F)$  and a Hopf algebra morphism  $\pi : H(F) \longrightarrow A$  such that  $(1_V \otimes \pi) \circ \beta_V = \alpha_V$ . In particular for every finite type cosovereign Hopf algebra  $(A, \Phi)$ , there is a surjective Hopf algebra morphism  $\pi : H(F) \longrightarrow A$  for some  $F \in GL_n(k)$ .

Theorem 4.2 justify the expression universal cosovereign Hopf algebras for the algebras  $H(F)$ .

**Proof.** It is easily seen that the maps  $\Delta$  and  $\varepsilon$  are well defined algebra morphisms and thus  $H(F)$  is a bialgebra. In the same way the formulas of the theorem give rise to a well defined anti-homomorphism  $S$  which is clearly an antipode for  $H(F)$ . Once again it is clear that  $S$  is bijective with inverse as given in the theorem. The character  $\Phi_F$  is easily seen to be well defined on  $H(F)$ . We have  $S^{-1}(u) = {}^tF^t v {}^tF^{-1} = \Phi_F(u)S(u)\Phi_F^{-1}(u)$  and  $S^{-1}(v) = {}^t u = F^{-1}F^t u F^{-1}F = F^{-1}S(v)F = \Phi_F(v)S(v)\Phi_F^{-1}(v)$ . Therefore  $S^{-1} = \Phi_F * S * \Phi_F^{-1}$  and  $(H(F), \Phi_F)$  is a cosovereign Hopf algebra.

Let  $A$  be a Hopf algebra with bijective antipode and let  $V$  be a finite dimensional  $A$ -comodule with basis  $v_1, \dots, v_n$  such that  $\alpha_V(v_i) = \sum_j v_j \otimes a_{ji}$ . We have  $V^\vee \cong {}^\vee V$  and hence there is a matrix  $F \in GL_n(k)$  such that  ${}^tS(a)F = F^t S^{-1}(a)$  ( $a$  is the matrix  $(a_{ij})$ ). Then there is a Hopf algebra morphism  $\pi : H(F) \longrightarrow A$  defined by  $\pi(u) = a$  and  $\pi(v) = {}^tS(a)$  (since  $({}^t a)^{-1} = {}^tS^{-1}(a)$ ). A coaction  $\beta_V : V \longrightarrow V \otimes H(F)$  is defined by  $\beta_V(v_i) = \sum_j v_j \otimes u_{ji}$ . It is clear that  $\pi$  satisfies the requirement in the theorem. The last assertion is straightforward.  $\square$

The Hopf algebras  $H(F)$  clearly show that anything can happen with the dimension theory of sovereign monoidal categories. Let  $U$  be the obvious  $n$ -dimensional comodule associated with the matrix  $u$ . Then  $\dim_{\Phi_F}^l(U) = \text{Tr}(F)$  and  $\dim_{\Phi_F}^r(U) = \text{Tr}(F^{-1})$ . These two scalars may not coincide. It is also clear in this example that the dimension of an object may take any value with respect to different sovereign structure.

The next result reduces the list of the algebras  $H(F)$ :

**Proposition 4.3** *Let  $F$  and  $K \in GL_n(k)$  and let  $\lambda \in k^*$ . Then  $H(\lambda F) = H(F)$ ,  $H(F) \cong H(KFK^{-1})$  and  $H(F) \cong H({}^tF^{-1})$  (as Hopf algebras).*

**Proof.** The first statement is clear. A Hopf algebra isomorphism  $\phi : H(F) \rightarrow H(KFK^{-1})$  is defined by  $\phi(u) = {}^tKu {}^tK^{-1}$  and  $\phi(v) = K^{-1}vK$ . A Hopf algebra isomorphism  $\psi : H(F) \longrightarrow H({}^tF^{-1})$  is defined by  $\psi(u) = v$  and  $\psi(v) = FuF^{-1}$ .  $\square$

We now study some cosemisimplicity properties of  $H(F)$ . Let us recall that a Hopf algebra is cosemisimple if and only if there is a Haar measure on it ([1]).

**Proposition 4.4** *Let  $F \in GL_n(k)$ . If  $\text{Tr}(F) = 0$  or  $\text{Tr}(F^{-1}) = 0$  then  $H(F)$  is not cosemisimple.*

**Proof.** One can assume that the base field is algebraically closed. The  $n$ -dimensional comodule  $U$  associated to the elements  $u_{ij}$  is clearly irreducible. The result follows from an application to  $U$  of Larson's orthogonality relation ([15]) as expressed by Woronowicz ([32], see also [8], proposition 3.5).  $\square$

We now make contact with the theory of compact quantum groups. We assume that the reader is familiar with Hopf  $*$ -algebras [29] and with the algebraic theory of compact quantum groups as in [8]. We now assume the base field to be the field of complex numbers.

**Notations.** Let  $a = (a_{ij})$  where  $A$  is a  $*$ -algebra. Then the matrix  $(a_{ij}^*)$  is denoted by  $\bar{a}$  and the matrix  ${}^t\bar{a}$  is denoted by  $a^*$ .

**Definition 4.5** A Hopf  $*$ -algebra is a complex algebra  $A$ , which is a  $*$ -algebra and whose coproduct  $\Delta : A \longrightarrow A \otimes A$  is a  $*$ -homomorphism. A  $CQG$  algebra is Hopf  $*$ -algebra  $A$  such that every finite-dimensional  $A$ -comodule is unitarizable. In other words for every matrix  $a = (a_{ij}) \in M_n(A)$  such that  $\Delta(a_{ij}) = \sum_k a_{ik} \otimes a_{kj}$  and  $\varepsilon(a_{ij}) = \delta_{ij}$ , there is a matrix  $F \in GL_n(\mathbb{C})$  such that the matrix  $FaF^{-1}$  is unitary.

A  $CQG$  algebra may be thought as the algebra of representative functions on a compact quantum group. A Hopf  $*$ -algebra is  $CQG$  if and only if there is a faithful Haar measure on it ([8], 3.10). In particular a  $CQG$  algebra is cosemisimple. In the next result we find a necessary and sufficient condition for the Hopf algebra  $H(F)$  to admit a  $CQG$  algebra structure. We say that a matrix  $F \in GL_n(\mathbb{C})$  is relatively positive if there is a scalar  $\lambda \in \mathbb{C}^*$  such that  $\lambda F$  is a positive matrix.

**Proposition 4.6** Let  $F \in GL_n(\mathbb{C})$ . Then  $H(F)$  admits a  $CQG$  algebra structure if and only if  $F$  is conjugate to a relatively positive matrix.

**Proof.** Let us assume that  $H(F)$  admits a  $CQG$  algebra structure. Then there is  $K \in GL_n(\mathbb{C})$  such that  $KuK^{-1}$  is unitary and hence  $uK^{-1}K^{-1*}u^*K^*K = I$  which implies  ${}^t v = K^{-1}K^{-1*}u^*K^*K$  and  ${}^t u = {}^t K \bar{K} v^* \bar{K}^{-1} {}^t K^{-1}$ . There is  $Q \in GL_n(\mathbb{C})$  such that  $QvQ^{-1}$  is unitary and hence  $vQ^{-1}Q^{-1*}v^*Q^*Q = I$  which implies  $F{}^t u F^{-1} = Q^{-1}Q^{-1*}v^*Q^*Q$  and  ${}^t u = F^{-1}Q^{-1}Q^{-1*}v^*Q^*QF$ . We get  $Q^*QF{}^t K \bar{K} = \alpha I$  for some  $\alpha \in \mathbb{C}^*$  (the  $v_{ij}$ 's are linearly independent). Therefore  ${}^t K^{-1}F{}^t K = \alpha {}^t K^{-1}Q^{-1}Q^{-1*}\bar{K}^{-1}$  and  $F$  is conjugate to a relatively positive matrix.

Conversely we can assume that  $F$  is a positive matrix by proposition 4.3. It is easy to see that a Hopf  $*$ -algebra structure is defined on  $H(F)$  by letting  $\bar{u} = v$  (at this point we only use  $F^* = F$ ). The matrix  $u$  is unitary. Let  $K = \sqrt{F}$ . It is easy to see that the matrix  $KvK^{-1}$  is unitary. It follows that  $H(F)$  is a  $CQG$  algebra since it is generated by the matrices  $u$  and  $v$  ([8], proposition 2.4).  $\square$

When  $F$  is a positive matrix, it is easy to see that  $H(F)$  is the  $CQG$  algebra of representative functions on the compact quantum group  $A_u(F)$  defined by Van Daele and Wang in theorem 1.3 of [30] (in fact it is sufficient to consider positive matrices to get the family of universal compact quantum groups). In that case the representation semi-ring of  $H(F)$  is described by Banica in [2]: the irreducible comodules are labelled by the free product  $\mathbb{N} * \mathbb{N}$ .

Woronowicz has shown ([32], theorem 5.6) that a  $CQG$  algebra always has a sovereign character. For a general cosemisimple Hopf algebra  $A$ , there is a convolution invertible  $\lambda$  such that  $S^2 = \lambda * id * \lambda^{-1}$  ([15]) and the map  $\sigma = \lambda * id * \lambda$  is an algebra morphism (the modular homomorphism of theorem 5.6 in [32]). However it does not seem to be clear that a cosemisimple Hopf algebra always has a sovereign character.

## 5 Other examples

We first take a look at finite-dimensional Hopf algebras.

**Theorem 5.1** *Let  $(A, \Phi)$  be a finite-dimensional cosovereign (or sovereign) Hopf algebra over a field of characteristic zero. If  $\dim_\Phi^l(A) \neq 0$  or  $\dim_\Phi^r(A) \neq 0$  then  $S \circ S = 1_A$  and  $A$  is semisimple and cosemisimple.*

**Proof.** We can assume that  $k$  is algebraically closed and that  $(A, \Phi)$  is a cosovereign Hopf algebra (the sovereign case is dual). We consider  $A$  as an  $A$ -comodule via the comultiplication. For every finite-dimensional  $A$ -comodule there is an  $A$ -comodule isomorphism  $A \otimes V \cong A^{\dim(V)}$  (see proposition 1 in [4]). By proposition 3.17 we have  $\dim_\Phi^l(A)\dim_\Phi^l(V) = \dim(V)\dim_\Phi^l(A)$  (the dimension is clearly additive on direct sums). In particular  $\dim_\Phi^l(A) = \dim(A)$  if  $\dim_\Phi^l(A) \neq 0$ . Let  $e_1, \dots, e_n$  be a basis of  $A$  such that  $\Delta(e_i) = \sum_j e_j \otimes a_{ji}$ . Let  $F$  be the matrix  $F = (\Phi(a_{ij}))$ . The matrix  $F$  can be assumed to be triangular. We have  $\Phi^{*k} = \varepsilon$  for some integer  $k$  since  $\Phi$  is a character and hence  $F^k = I$ . This means that the elements  $\Phi(a_{ii})$  are  $k$ -th roots of unity. But  $\dim_\Phi^l(A) = \dim(A) = n = \sum_i \Phi(a_{ii})$  and therefore  $\Phi(a_{ii}) = 1$  for all  $i$  since the base field is of characteristic 0. This also implies that  $F$  is a diagonal matrix and  $F = I$ . Then  $\Phi = \varepsilon$  and  $S \circ S = 1_A$ . Now  $A$  is semisimple by [15] theorem 4.3 and is also cosemisimple by [16]. The proof is the same if  $\dim_\Phi^r(A) \neq 0$ .  $\square$

**Example 5.2** It is easy to see that Sweedler's famous 4-dimensional Hopf algebra (see [14], p. 67) admits a sovereign element (the group-like element) and a sovereign character (the only non-trivial character). In both cases, the left and right dimensions of this algebra are equal to zero.

Sweedler's algebra clearly shows a way to construct sovereign Hopf algebras. Let  $H_n$  be the quotient of the free algebra  $k\{X_1, \dots, X_n, \Phi, \Phi^{-1}\}$  by the two-sided ideal generated by the relations  $\Phi\Phi^{-1} = 1 = \Phi^{-1}\Phi$ . Then  $H_n$  is a Hopf algebra with comultiplication  $\Delta(X_i) = 1 \otimes X_i + X_i \otimes \Phi$ ,  $\Delta(\Phi) = \Phi \otimes \Phi$ , with counit  $\varepsilon(X_i) = 0$ ,  $\varepsilon(\Phi) = 1$  and with antipode  $S(X_i) = X_i\Phi$  and  $S(\Phi) = \Phi^{-1}$ . It is easy to see that  $S$  is bijective and that  $\Phi$  is a sovereign element in  $H_n$ . For more examples of this kind, see [26, 19].

We now examine a class of examples closely related to the quantum groups  $SU(n)$  of [33].

Let  $V = k^n$  and let  $e_1, \dots, e_n$  be the canonical basis with dual basis  $e_1^*, \dots, e_n^*$ . Let  $N \geq 2$  be an integer and let  $E : V^{\otimes N} \rightarrow k$  be a linear map. Let  $E(i_1, \dots, i_N) = E(e_{i_1} \otimes \dots \otimes e_{i_N})$ . We say that  $E$  is left non-degenerate if the linear map

$$V^{\otimes N-1} \rightarrow V^*, \quad e_{i_1} \otimes \dots \otimes e_{i_{N-1}} \mapsto \sum_k E(i_1, \dots, i_{N-1}, k) e_k^*$$

is surjective. In this case there are scalars  $\lambda(i_1, \dots, i_N)$  such that

$$(\star) \quad \sum_{j_1, \dots, j_{N-1}} \lambda(i, j_1, \dots, j_{N-1}) E(j_1, \dots, j_{N-1}, k) = \delta_{ik}, \quad 1 \leq i, k \leq n.$$



We say that  $E$  is right non-degenerate if the linear map

$$V^{\otimes N-1} \longrightarrow V^*, \quad e_{i_1} \otimes \dots \otimes e_{i_{N-1}} \longmapsto \sum_k E(k, i_1, \dots, i_{N-1}) e_k^*$$

is surjective. In that case there are scalars  $\mu(i_1, \dots, i_N)$  such that

$$(\star\star) \quad \sum_{j_1, \dots, j_{N-1}} E(k, j_1, \dots, j_{N-1}) \mu(j_1, \dots, j_{N-1}, i) = \delta_{ik}, \quad 1 \leq i, k \leq n.$$

**Theorem 5.3** *Let  $E : V^{\otimes N} \longrightarrow k$  be a left and right non-degenerate linear map. Let  $SL(E)$  be the universal algebra with generators  $(a_{ij})_{1 \leq i, j \leq n}$  and relations:*

$$(5.3.1) \quad \sum_{j_1, \dots, j_N} E(j_1, \dots, j_N) a_{j_1 i_1} \dots a_{j_N i_N} = E(i_1, \dots, i_N) 1, \quad 1 \leq i_1, \dots, i_N \leq n$$

$$(5.3.2) \quad \sum_{j_1, \dots, j_N} E(j_1, \dots, j_N) a_{i_1 j_1} \dots a_{i_N j_N} = E(i_1, \dots, i_N) 1, \quad 1 \leq i_1, \dots, i_N \leq n$$

i) *Then  $SL(E)$  is a Hopf algebra with bijective antipode.*

ii) *Assume that there are invertible scalars  $(\beta_i)_{1 \leq i \leq n}$  such that  $E(j_1, \dots, j_{N-1}, i) = \beta_i E(i, j_1, \dots, j_{N-1})$  for all  $i, j_1, \dots, j_{N-1}$ . Then there is a sovereign character  $\Phi_\beta$  on  $SL(E)$  such that  $\Phi_\beta(a_{ij}) = \delta_{ij} \beta_i$ .*

iii) *If  $k$  is a field of characteristic zero and if the field  $\mathbb{Q}(E(j_1, \dots, j_N)_{1 \leq i_1, \dots, i_N \leq n})$  can be ordered, then  $SL(E)$  is cosemisimple.*

iv) *If  $k = \mathbb{C}$  and  $E(j_1, \dots, j_N) \in \mathbb{R}$  for all  $j_1, \dots, j_N$ , then  $SL(E)$  admits a CQG algebra structure.*

**Proof.** i) It is easily seen that  $SL(E)$  is a bialgebra with coproduct  $\Delta(a_{ij}) = \sum_k a_{ik} \otimes a_{kj}$  and counit  $\varepsilon(a_{ij}) = \delta_{ij}$ . Let us show that the matrix  $a = (a_{ij})$  is invertible. Let us consider equation 5.3.1. Multiplying by  $\lambda(k, i_1, \dots, i_{N-1})$  and summing over  $i_1, \dots, i_{N-1}$ , we get that  $a$  is left invertible (we use  $(\star)$ ). In the same way  $a$  is right invertible (use 5.3.2 and  $(\star\star)$ ) and therefore  $a$  is invertible and  $SL(E)$  is a Hopf algebra by [31], theorem 1. Let us show that  ${}^t a$  is invertible. By 5.3.2 and  $(\star)$   ${}^t a$  is right invertible and by 5.3.1 and  $(\star\star)$   ${}^t a$  is left invertible. Hence the antipode of  $SL(E)$  is invertible.

ii) The character  $\Phi_\beta$  is easily seen to be well defined. Let us consider equation 5.3.2: multiplying on the left by  $S(a_{ki_1})$  and summing over  $i_1$ , we get

$$(\star\star\star) \quad \sum_{j_2, \dots, j_N} E(k, j_2, \dots, j_N) a_{i_2 j_2} \dots a_{i_N j_N} = \sum_{i_1} S(a_{ki_1}) E(i_1, \dots, i_N).$$

We have

$$\begin{aligned}
& \sum_i \sum_k E(k, i_2, \dots, i_N) S(a_{ik}) \beta_i a_{ji} \\
&= \sum_i \sum_{j_2, \dots, j_N} E(i, j_2, \dots, j_N) \beta_i a_{i_2 j_2} \dots a_{i_N j_N} a_{ji} \quad \text{by } (\star \star \star) \\
&= \sum_i \sum_{j_2, \dots, j_N} E(j_2, \dots, j_N, i) a_{i_2 j_2} \dots a_{i_N j_N} a_{ji} \\
&= E(i_2, \dots, i_N, j) \quad (\text{by 5.3.2}) = \beta_j E(j, i_2, \dots, i_N).
\end{aligned}$$

Using  $(\star \star)$  we get

$$\sum_i S(a_{il}) \beta_i a_{ji} = \delta_{jl} \beta_j \quad \text{for all } j, l.$$

The inverse of the matrix  ${}^t a$  is  ${}^t S^{-1}(a)$  and hence we have  $S^{-1}(a_{il}) = \beta_i S(a_{il}) \beta_j^{-1} = \Phi_\beta * S * \Phi_\beta^{-1}(a_{il})$ . This means that  $\Phi_\beta$  is a sovereign character on  $SL(E)$ .

iii) There is an algebra automorphism  $\tau$  of  $SL(E)$  defined by  $\tau(a_{ij}) = a_{ji}$ . Hence by [3], 4.7  $SL(E)$  is cosemisimple. Statement iv) follows from [3], 4.6.  $\square$

In a special case the  $SL(E)$  construction leads to the quantum groups  $SL_q$ . Let  $N = n$  and let  $q \in k^*$ . Let  $E_q : V^{\otimes n} \longrightarrow k$  defined by  $E_q(i_1, \dots, i_N) = 0$  if two indices are equal and otherwise  $E_q(i_1, \dots, i_N) = (-q)^{l(\sigma)}$  where  $l(\sigma)$  is the length of the permutation  $\sigma(k) = i_k$ . The Hopf algebras  $SL(E_q)$  and  $SL_q(n)$  are isomorphic. This fact can be proved using the same proof as in Rosso's comparison of the quantum  $SL$  groups of Woronowicz ([33]) and Drinfeld ([9]) ([21], theorem 6). See also [29] for useful computations. The elements  $\beta_i$  of theorem 5.3 are given by  $\beta_i = (-q)^{n+1-2i}$  and therefore  $SL_q(n)$  admits a sovereign character  $\Phi$  defined by  $\Phi(a_{ij}) = \delta_{ij} (-q)^{n+1-2i}$ .

## A Appendix

In this appendix we explicitly write and prove the correspondence between cotwists and sovereign characters for a cobarred Hopf algebra. We intensively use Sweedler's notations [25] : if  $a$  is an element of a Hopf algebra we write  $\Delta(a) = \sum a_1 \otimes a_2$ . We first recall some basic definitions (see [7, 14, 12]).

**Definition A.1** *A cobarred Hopf algebra is a pair  $(A, \sigma)$  where  $A$  is a Hopf algebra and  $\sigma : A \otimes A \longrightarrow k$  is a convolution invertible linear map satisfying:*

- (A.1)  $\sigma * m = m^{\text{op}} * \sigma$ , ie for all  $x, y \in A$  we have :  $\sum \sigma(x_1, y_1) x_2 y_2 = \sum y_1 x_1 \sigma(x_2, y_2)$ .
- (A.2)  $\sigma(xy, z) = \sum \sigma(x, z_1) \sigma(y, z_2)$  for all  $x, y, z \in A$ .
- (A.3)  $\sigma(x, yz) = \sum \sigma(x_1, z) \sigma(x_2, y)$  for all  $x, y, z \in A$ .

*A cotwist on a cobarred Hopf algebra  $(A, \sigma)$  is a central convolution invertible linear form  $\tau$  on  $A$  satisfying  $\tau \circ m = {}^t \sigma * (\tau \otimes \tau) * \sigma$  (where  ${}^t \sigma(x, y) = \sigma(y, x)$ ), ie for all  $x, y \in A$  we have  $\tau(xy) = \sum \sigma(y_1, x_1) \tau(x_2) \tau(y_2) \sigma(x_3, y_3)$ .*

Let  $\sigma^{-1}$  be the convolution inverse of  $\sigma$ . The following equalities hold for all  $x, y, z \in A$  (see [7]):

$$(A'.1) \sum \sigma^{-1}(x_1, y_1) y_2 x_2 = \sum x_1 y_1 \sigma^{-1}(x_2, y_2).$$

$$(A'.2) \sigma^{-1}(xy, z) = \sum \sigma^{-1}(y, z_1) \sigma^{-1}(x, z_2)$$

$$(A'.3) \sigma^{-1}(x, yz) = \sum \sigma^{-1}(x_1, y) \sigma^{-1}(x_2, z)$$

$$(A.4) \sigma(1, x) = \varepsilon(x) = \sigma(x, 1) ; (A'.4) \sigma^{-1}(1, x) = \varepsilon(x) = \sigma^{-1}(x, 1)$$

$$(A.5) \sigma^{-1}(x, y) = \sigma(S(x), y) ; (A.6) \sigma(x, y) = \sigma^{-1}(x, S(y)) ; (A.7) \sigma(x, y) = \sigma(S(x), S(y)).$$

Let  $\lambda$  be the linear form on  $A$  defined by  $\lambda(x) = \sum \sigma(x_1, S(x_2))$ . Then  $\lambda$  is invertible with inverse given by  $\beta(x) = \sum \sigma^{-1}(S(x_1), x_2)$ . Furthermore  $S^2 = \beta * id * \lambda$  ([7], theorem 1.3) and in particular the antipode of a cobraided Hopf algebra is bijective.

We first observe the following result:

**Lemma A.2** *Let  $(A, \sigma)$  be a cobraided Hopf algebra. Then  $\beta$  satisfies the cotwist equation  $\beta \circ m = {}^t\sigma * (\beta \otimes \beta) * \sigma$ .*

**Proof.** Let  $x, y \in A$ . We have

$$\begin{aligned} & {}^t\sigma^{-1} * (\beta \circ m) * \sigma^{-1}(xy) \\ &= \sum \sigma^{-1}(y_1, x_1) \sigma^{-1}(S(x_2 y_2), x_3 y_3) \sigma^{-1}(x_4, y_4) \\ &= \sum \sigma^{-1}(y_2, x_2) \sigma^{-1}(S(y_1 x_1), x_3 y_3) \sigma^{-1}(x_4, y_4) \quad (\text{by } \sigma^{-1} * m^{\text{op}} = m * \sigma^{-1}) \\ &= \sum \sigma^{-1}(y_2, x_2) \sigma^{-1}(S(y_1), x_3 y_3) \sigma^{-1}(S(x_1), x_4 y_4) \sigma^{-1}(x_5, y_5) \quad (\text{by (A'.2)}) \\ &= \sum \sigma^{-1}(y_3, x_2) \sigma^{-1}(S(y_2), x_3) \sigma^{-1}(S(y_1), y_4) \sigma^{-1}(S(x_1), x_4 y_5) \sigma^{-1}(x_5, y_6) \quad (\text{by (A'.3)}) \\ &= \sum \sigma^{-1}(S(y_1), y_2) \sigma^{-1}(S(x_1), x_2 y_3) \sigma^{-1}(x_3, y_4) \quad (\text{by (A'.2) and (A'.4)}) \\ &= \sum \beta(y_1) \sigma^{-1}(S(x_1), x_2 y_2) \sigma^{-1}(x_3, y_3) \\ &= \sum \beta(y_1) \sigma^{-1}(S(x_1), y_3 x_3) \sigma^{-1}(x_2, y_2) \quad (\text{by } m * \sigma^{-1} = \sigma^{-1} * m^{\text{op}}) \\ &= \sum \beta(y_1) \sigma^{-1}(S(x_2), y_3) \sigma^{-1}(S(x_1), x_4) \sigma^{-1}(x_3, y_2) \quad (\text{by (A'.3)}) \\ &= \sum \beta(y_1) \sigma^{-1}(S(x_2) x_3, y_2) \sigma^{-1}(S(x_1), x_4) \quad (\text{by (A'.2)}) \\ &= \beta(x) \beta(y) \quad \text{by (A'.4)}. \quad \square \end{aligned}$$

On the Hopf algebra level, proposition 2.11 from [34] takes the following form:

**Theorem A.3** *Let  $(A, \sigma)$  be a cobraided Hopf algebra. There is a bijective correspondence between sovereign characters on  $A$  and cotwists on  $(A, \sigma)$ . Explicitly we have:*

- i) *If  $\Phi$  is a sovereign character on  $A$  then  $\tau = \Phi * \beta$  is a cotwist on  $(A, \sigma)$ .*
- ii) *If  $\tau$  is a cotwist on  $(A, \sigma)$  then  $\Phi = \tau * \beta^{-1}$  is a sovereign character on  $A$*

**Proof.** i) Let us first show that  $\tau$  is central. For this purpose it is sufficient to prove that  $id * \tau = \tau * id$ . We have  $id * \tau = id * \Phi * \beta = \Phi * S^2 * \beta = \Phi * \beta * id = \tau * id$  since  $S^2 * \beta = \beta * id$  and  $\Phi * S^2 = id * \Phi$ . Let  $x, y \in A$ . We have

$$\begin{aligned}
\tau(xy) &= \sum \Phi(x_1)\Phi(y_1)\beta(x_2y_2) \quad (\text{since } \Phi \text{ is a character}) \\
&= \sum \Phi(x_1)\Phi(y_1)\sigma(y_2, x_2)\beta(x_3)\beta(y_3)\sigma(x_4, y_4) \quad (\text{by lemma A.2}) \\
&= \sum \sigma(S^{-2}(y_1), S^{-2}(x_1))\Phi(x_2)\Phi(y_2)\beta(x_3)\beta(y_3)\sigma(x_4, y_4) \quad (\Phi * id = S^{-2} * \Phi) \\
&= \sum \sigma(y_1, x_1)\Phi(x_2)\beta(x_3)\Phi(y_2)\beta(y_3)\sigma(x_4, y_4) \quad (\text{by (A.7)}) \\
&= {}^t\sigma * (\tau \otimes \tau) * \sigma(x, y).
\end{aligned}$$

Therefore  $\tau$  is a cotwist on  $(A, \sigma)$ .

ii) Let us first remark that  $\beta^{-1}(x) = \sum \sigma(x_1, S(x_2)) = \sum \sigma(S^{-1}(x_1), x_2)$  by (A.7) and that  $\Phi = \beta^{-1} * \tau$  since  $\tau$  is central. Let  $x, y \in A$ . We have

$$\begin{aligned}
\Phi(xy) &= \sum \beta^{-1}(x_1y_1)\tau(x_2y_2) \\
&= \sum \sigma(S^{-1}(y_1)S^{-1}(x_1), x_2y_2)\tau(x_3y_3) \\
&= \sum \sigma(S^{-1}(y_1), x_2y_2)\sigma(S^{-1}(x_1), x_3y_3)\tau(x_4y_4) \quad (\text{by (A.2)}) \\
&= \sum \sigma(S^{-1}(y_2), y_3)\sigma(S^{-1}(y_1), x_2)\sigma(S^{-1}(x_1), x_3y_4)\tau(x_4y_5) \quad (\text{by (A.3)}) \\
&= \sum \beta^{-1}(y_2)\sigma(S^{-1}(y_1), x_2)\tau(x_3y_3)\sigma(S^{-1}(x_1), x_4y_4) \quad (\tau \text{ central}) \\
&= \sum \beta^{-1}(y_1)\sigma(S(y_2), x_2)\tau(x_3y_3)\sigma(S^{-1}(x_1), x_4y_4) \quad (S^{-1} * \beta^{-1} = \beta^{-1} * S) \\
&= \sum \beta^{-1}(y_1)\sigma^{-1}(y_2, x_2)\tau(x_3y_3)\sigma(S^{-1}(x_1), x_4y_4) \quad (\text{by (A.5)}) \\
&= \sum \beta^{-1}(y_1)\tau(x_2)\tau(y_2)\sigma(x_3, y_3)\sigma(S^{-1}(x_1), x_4y_4) \quad (\tau \text{ twist}) \\
&= \sum \beta^{-1}(y_1)\tau(x_3)\tau(y_2)\sigma(x_4, y_3)\sigma(S^{-1}(x_2), y_4)\sigma(S^{-1}(x_1), x_5) \quad (\text{by (A.3)}) \\
&= \sum \beta^{-1}(y_1)\tau(x_4)\tau(y_2)\sigma(x_3, y_3)\sigma(S^{-1}(x_2), y_4)\sigma(S^{-1}(x_1), x_5) \quad (\tau \text{ central}) \\
&= \sum \beta^{-1}(y_1)\tau(x_4)\tau(y_2)\sigma(x_3S^{-1}(x_2), y_3)\sigma(S^{-1}(x_1), x_5) \quad (\text{by (A.2)}) \\
&= \sum \beta^{-1}(y_1)\tau(x_2)\tau(y_2)\sigma(S^{-1}(x_1), x_3) \quad (\text{by (A.4)}) \\
&= \sum \beta^{-1}(y_1)\tau(y_2)\tau(x_3)\sigma(S^{-1}(x_1), x_2) \quad (\tau \text{ central}) \\
&= \sum \beta^{-1}(y_1)\tau(y_2)\beta^{-1}(x_1)\tau(x_2) = \Phi(x)\Phi(y).
\end{aligned}$$

Therefore  $\Phi$  is a character. Now  $\Phi * S * \Phi^{-1} = \beta^{-1} * \tau * S * \tau^{-1} * \beta = \beta^{-1} * S * \beta = S^{-1}$  ( $\tau$  is central):  $\Phi$  is a sovereign character.  $\square$

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